Quasiradiation solution to the compound integrable model

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We introduce compound integrable models composed of different systems of nonlinear equations describing evolution of fields and matter in different space and time intervals. As an example, we investigate the integrable compound model, which includes the modified nonlinear Schrödinger equation describing propagation of ultrashort pulses in an optical fiber and system of equations describing the two-wave mixing in a resonant medium with the two-photon induced Kerr-type nonlinearity. Using the matrix Riemann-Hilbert factorization approach for nonlinear evolution equations integrable in the sense of the the inverse scattering method, we study generation of ultrashort pulses in this model. We find a solution of a spectral problem on the semi-infinite interval and solve the compound model for simple but nontrivial boundary conditions for the resonant medium. We show that an asymptotic solution for light pulse propagating in the fiber is described by the quasiradiation solution to the modified nonlinear Schrödinger equation.

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I. INTRODUCTION

The study of the fundamental dynamical processes associated with the propagation of high-power ultrashort pulses in optical fibers is of paramount importance: the nonlinear soliton(ic), or near-soliton(ic), operating mode(s) in such systems are very promising; in particular, very high data transmission rates, high noise immunity, and the accessibility of new frequency bands [1]. The classical, mathematical model for nonlinear pulse propagation in the picosecond time scale in the anomalous dispersion regime in an isotropic, homogeneous, lossless, nonamplifying, polarization-preserving single-mode optical fiber is the nonlinear Schrödinger equation (NLSE) [1]; however, in the subpicosecond-femtosecond time scale, experiments and theories on the propagation of high-power ultrashort pulses in long monomode optical fibers have shown that the NLSE is no longer valid and that additional nonlinear terms (dispersive and dissipative) and higher-order linear dispersion should be taken into account [1]. In this case, subpicosecond-femtosecond pulse propagation is described (in dimensionless and normalized form) by the following nonlinear evolution equation (MNLSE),

$$i\partial_\xi u+\tfrac{1}{2}\partial_\tau^2 u+|u|^2u+is\partial_\tau(|u|^2u)=\mathcal{R}, \qquad (1.1)$$

where u is the slowly varying amplitude of the complex field envelope, ξ is the propagation distance along the fiber length, τ is the time measured in a frame of reference moving with the pulse at the group velocity (the retarded frame), $s(>0)$ governs the effects due to the intensity dependence of the group velocity (self steepening). It has been shown recently [1] that the MNLSE utilizing the notion of slowly varying envelope is still valid up to 3–5 periods of field oscillations within the envelope. Term $\mathcal{R}$ in the right-hand side of Eq. (1.1) may include the terms that governs, for instance, the soliton self-frequency shift effect, the Raman gain, the intrinsic fiber loss, and the effects of resonant n-photon interaction of auxiliary fields with the ''main'' field u [1]. In the most theoretical studies, such terms are treated as the perturbations having a small influence on evolution of the main field and contribution of these terms are usually investigated analytically by using the perturbation theory. Such an approach is mainly used for the study of change of the form and parameters of propagating stable pulse having initially the form of soliton due to these perturbations. However, for a strong enough influence of the terms composing $\mathcal{R}$, the perturbation approaches are not applicable. Besides, interaction with the auxiliary fields may yield the generation of the additional ultrashort pulses and crucially change asymptotic behavior of the main field.

The versions of the inverse scattering method had been developed only for one space and time interval for media with homogeneous nonlinear and dispersive properties [2]. However, being applied in the nonlinear optics, this method can be generalized to the case of interaction of fields with matter composed of the different nonlinear media situated in the different space intervals. From another side, interaction of the fields with the same media may be different in the different time intervals. Consider, e.g., a two-level optical medium permitted as one-photon and two-photon transitions. Let a light field interact with such a medium under the condition of one-photon resonance during some time interval, and then a couple of fields being injected in the same media interact with the same transition under the condition of the two-photon resonance during the next time interval.

In practice, optical fiber is used as a part of a complex experimental optical setup that includes different nonlinear and linear auxiliary media as well. Interaction of fields in these auxiliary media may be used for the generation of the ultrashort pulses, amplification, selection of pulses, and so on. Then these pulses being injected in the fiber are used for information transfer and others aims. In this case, in the right-hand side of Eq. (1.1), $\mathcal{R}$ may include interaction of the main field u with auxiliary fields and others effects in the space intervals out of the fiber. Therefore, interaction pro-

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cesses are spatially separated and one process affects the other process occurring in the next medium via the boundary conditions.

The study of such schemes of interaction leads to an important class of nonlinear integrable problems that have both an initial value and also a nontrivial boundary value. Of these, the simplest ones are those of an hyperbolic form, such as the sine-Gordon equation and others of that structure. For these equations, the initial value—the boundary-value problem is a well-stated problem, and the existence and uniqueness of the solution is well established. These equations are also integrable. Thus, one can also analyze these problems and study their solution by the use of an inverse scattering transform (ISTM) [2] on a finite or semi-infinite interval. Such an ISTM requires the solution of a direct and inverse scattering problem on the corresponding interval. Mentioned works [3–8] have described various formal aspects of this problem. Some papers [4,5] have dealt with rather special boundary problems that are impractical to be generalized. As first pointed out in Ref. [3], and as later detailed in Ref. [6], the evolution of the scattering data along a boundary where there is boundary data, can strongly differ from what we are familiar with in the infinite interval. The ISTM formalism based on the Reimann-Hilbert (RH) problem associated with the Zakharov *et al.* [2] linear spectral problem was used in Ref. [8] for the solution of the stimulated Raman scattering in semi-finite interval. Fokas [9] showed how the inverse scattering approach could be exploited to solve linear and nonlinear problems with nontrivial boundary and initial conditions. In this regard, the formulation in terms of Lax pairs proved to be invaluable, especially the realization by Fokas that the two Lax equations, when analyzed simultaneously and when supplemented with the analysis of a global relation, enabled solutions to be obtained by the Riemann-Hilbert or d-bar methods. His approach is valid for a general form of initial-boundary conditions and can be used for the semiaxis or finite interval problems.

For the compound models describing the interaction of fields and matter in the sequentially placed media with the different nonlinear properties the boundary-condition problem, in common, becomes more complex. In this case, the evolution of fields in one medium determines the boundary conditions for the next medium situated in the neighborhood in the direction of the fields propagation.

Development of analytical methods of solution of compound models describing the evolution of ultrashort light pulses in the compound medium including different finite or semi-infinite media are of practical interest.

The ultimate goals of this paper are: (i) the introduction of the integrable ''compound models'' that can be used for the study of pulse dynamics in a compound optical experimental setup, (ii) the construction of two new integrable models including one physical example of the compound model, and (iii) the study of the peculiarities of light field generation in such compound models for the nontrivial boundary conditions.

It is known that some waves mixing in medium with the two-photon induced Kerr-type nonlinearities (resonant medium) may be described by using the integrable models [10,11]. Here we find that under a set of approximations, the Maxwell equations are reduced to a couple of equations that are formally equivalent to a system of the second-harmonic generation with additional cubic terms describing the nonlinear Stark effects. The Lax representation for this integrable model is derived here. Then we construct the new integrable system of equations that is compound from the one above and the MNLSE describing evolution in the different space intervals.

This compound model describes the following physical situation. Let the compound medium be composed of a resonant medium situated in the space interval $[z_1=0,z_2]$ and optical fibers (Kerr medium) situated in the semi-infinite interval $[z_2,\infty]$. Initially, the main field is absent and the auxiliary fields are constant in time and space and nonzero in the resonant medium. Then the main field is generated from noise due to parametric interaction with the auxiliary fields. We show that for the long enough effective length of the resonant medium asymptotic solution to the compound model for the main field envelope is described by the quasiradiation solution of the MNLSE.

The approach for solution of the initial-boundary problem used here differs from that of Ref. [9]. This approach is more convenient in our opinion for the description of fields dynamics in the compound model chosen below for particular initial-boundary conditions.

The paper is organized as follows. In Sec. II, we describe a common structure of a compound model and the peculiarities of evolution of scattering data associated with such a model. Section III is devoted to the derivation of the compound model out of the physical onset. In Sec. IV the inverse scattering problem is applied to an integrable version of the compound model as the matrix oscillatory Riemann-Hilbert (RH) factorization problem. In Sec. V we show that the quasiradiation solution to the MNLSE that governs the ultrashort pulse evolution in optical fiber can be a corollary of the simple but nontrivial initial-boundary conditions for an auxiliary field.

II. COMPOUND INTEGRABLE MODELS

A. A common model

Here we describe structure of a common integrable model, which is compounded from $N\times M$ integrable models. Some of them may coincide. Assume that the compound model has the following Lax representation:

$$\frac{\partial}{\partial\tau}\psi(\tau,z;\lambda)=\sum_{j=1}^{M}\beta_{j,j+1}(\tau)L_j(\tau,z;\lambda)\psi(\tau,z;\lambda)\equiv\mathcal{L}\psi, \tag{2.1}$$

$$\frac{\partial}{\partial z}\psi(\tau,z;\lambda)=\sum_{i=1}^{N}\alpha_{i,i+1}(z)A_i(\tau,z;\lambda)\psi(\tau,z;\lambda)\equiv\mathcal{A}\psi; \tag{2.2}$$

$$\beta_{j,j+1}(\tau)=[\theta(\tau-\tau_j)\theta(-\tau+\tau_{j+1})]\widetilde{\beta_j}(\tau),\quad \tau_{j+1}>\tau_j;$$

$$\alpha_{i,i+1}(z)=[\theta(z-z_i)\theta(-z+z_{i+1})]\widetilde{\alpha_i}(z),\quad z_{i+1}>z_i,$$

where $\theta(z)$ is the step function: $\theta(z)=0,\ z\leqslant 0;\ \theta(z)=1,\ z>0$. $\widetilde{\beta_i}(\tau)\neq 0$, $\widetilde{\alpha_i}(z)\neq 0$ are the finite functions. $\alpha_{i,i+1}(z)/\widetilde{\alpha_i}(z)$, $\beta_{i,i+1}(\tau)/\widetilde{\beta_i}(\tau)$ are the projectors, i.e., $\alpha^2_{i,i+1}(z)/\widetilde{\alpha_i}^2(z)=\alpha_{i,i+1}(z)/\widetilde{\alpha_i}(z)$ and so on.

The compatibility condition of linear systems (2.1) and (2.2) is

$$\frac{\partial}{\partial z}\sum_{j=1}^{M}\beta_{j,j+1}(\tau)L_j-\frac{\partial}{\partial \tau}\sum_{i=1}^{N}\alpha_{i,i+1}(z)A_i+\left[\sum_{j=1}^{M}\beta_{j,j+1}(\tau)L_j,\sum_{i=1}^{N}\alpha_{i,i+1}(z)A_i\right]=0. \quad (2.3)$$

Multiplying (2.3) by $\alpha_{i,i+1}\beta_{j,j+1}\widetilde{\alpha}_i^{-1}\widetilde{\beta}_j^{-1}$ we derive

$$\widetilde{\beta}_j(\tau)\frac{\partial}{\partial z}L_j-\widetilde{\alpha_i}(z)\frac{\partial}{\partial \tau}A_i+[\widetilde{\beta}_j(\tau)L_j,\widetilde{\alpha_i}(z)A_i]=0;$$

$$z\in[z_i,z_{i+1}],\quad \tau\in[\tau_j,\tau_{j+1}]. \quad (2.4)$$

Consequently, an evolution in the space-time rectangle $[z_i,z_{i+1}],[\tau_j,\tau_{j+1}]$ is described by the evolution equations associated with the Lax representation: $\partial_\tau\psi=\widetilde{\beta}_jL_j\psi$, $\partial_z\psi=\widetilde{\alpha}_iA_i\psi$.

The ISTM technique for spectral problem (2.1) for more than one interval $[\tau_j,\tau_{j+1}]$, $\beta_{j,j+1}(\tau)\neq 0$ does not exist, as it is known for us. Below we will study a compound model associated with the Lax pair (2.1, 2.2) for only one semi-infinite interval $[\tau_1=0,\tau_2=\infty)$, $\beta_{1,2}\neq 0$. Note that a more common case of the compound model may have the physical application.

1. z-dependence of the scattering data

Consider a case of one interval $[\tau_1=0,\tau_2)$, $\tau_2\to\infty$ and N intervals $[z_i,z_{i+1}]$, $\alpha_{i,i+1}\neq 0$. Find z dependence of the scattering data for the simple ''boundary'' value $L_1(\tau,0;\lambda)=\text{const}$, i.e., for constant ''potential.'' Let two solutions ψ,ϕ to linear system (2.1) have the following values at $\tau=0,\infty$:

$$\psi(0,z;\lambda)=\Phi_0(\lambda),\quad \lim_{\tau_2\to\infty}\psi(\tau_2,z;\lambda)=\Phi_+(\lambda),$$

$$\Phi_+(\lambda)=e^{-i\sigma_3\nu_0}\Phi_0(\lambda)e^{i\sigma_3\nu_0}, \quad (2.5)$$

where ν_0 is an arbitrary real constant, $\sigma_3=\text{diag}(1,-1)$ is the Pauli matrix. For an example of a compound model considered in the present paper we have $\Phi_0=\mathbf{I}$. These solutions are related by the matrix $T(z;\lambda)$

$$\psi(\tau,z;\lambda)=\phi(\tau,z;\lambda)T(z;\lambda). \quad (2.6)$$

Substituting Eq. (2.6) into Eq. (2.1) we find

$$\frac{\partial}{\partial z}T(z;\lambda)=\widetilde{A}^+T(z;\lambda)-T(z;\lambda)\widetilde{A}, \quad (2.7)$$

here

$$\widetilde{A}^+=\Phi_+^{-1}(\lambda)\mathcal{A}(\tau_2,z;\lambda)\Phi_+(\lambda),$$

$$\widetilde{A}=\Phi_0^{-1}(\lambda)\mathcal{A}(0,z;\lambda)\Phi_0(\lambda)$$

.

Formal solution to Eq. (2.7) is

$$\begin{aligned}T_{i+1}(z;\lambda)&=\exp[\mathcal{B}(\tau_2,z)]T_1(0,\lambda)\exp[-\mathcal{B}(0,z)]\\&=\exp[\mathcal{B}_i(\tau_2,z)]T_i(z_i,\lambda)\exp[-\mathcal{B}_i(0,z)]\\&=\exp[\mathcal{B}_i(\tau_2,z)],\ldots,\exp[\mathcal{B}_1(\tau_2,z)]T_1(z_1;\lambda)\\&\quad\times\exp[-\mathcal{B}_1(0,z)],\ldots,\exp[-\mathcal{B}_i(0,z)],\end{aligned} \quad (2.8)$$

where we have introduced the functions

$$\mathcal{B}(y,z)=\int_0^z E^{-1}(y;\lambda)\sum_{i=1}^{N}\alpha_i(z)A_i(y,z;\lambda)E(y;\lambda),$$

$$\mathcal{B}_i(y,z)=\int_{z_i}^z E^{-1}(y;\lambda)A_i(y,z;\lambda)E(y;\lambda),\quad z_i<z\leqslant z_{i+1}.$$

From Eq. (2.8) it follows that the dependence T on z within the interval $[z_i,z_{i+1}]$ is determined by Eq. (2.7) with the boundary-value $T(z_i,\lambda)$.

For 2×2 matrices A_i,L_j such that $(A_i)_{12}(\tau_2,z;\lambda)=(A_i)_{21}(\tau_2,z;\lambda)=0$, for $\tau_2\to\infty$ the z dependence of matrix $T(z;\lambda)$ is governed by the equation

$$\frac{\partial}{\partial z}\mathcal{T}_{1,k}(z;\lambda)=-\mathcal{T}_{1,k}(z;\lambda)\widetilde{A}_k(0,z;\lambda), \quad (2.9)$$

where

$$\mathcal{T}_{1,k}=\lim_{\tau_2\to\infty}\exp\left[-i\sigma_3\int_0^z\sum_{i=1}^{k}\alpha_{i,i+1}(s)\times(\widetilde{A}_i)_{11}(\tau_2,s;\lambda)ds\right]T_{1,k},$$

$$\widetilde{A}_k(0,z;\lambda)=\Phi_0^{-1}(\lambda)A_k(0,z;\lambda)\Phi_0(\lambda)$$

.

Formal solution to Eq. (2.9) is

$$\begin{aligned}\mathcal{T}_{1,N}&=\mathcal{T}_{1,2}\mathcal{T}_{2,3},\ldots,\mathcal{T}_{N-1,N},\mathcal{T}_{i,i+1}(z;\lambda)\\&=\exp\left[\int_{z_i}^{z_{i+1}}\alpha_{i,i+1}(s)\widetilde{A}_i(0,s;\lambda)ds\right].\end{aligned} \quad (2.10)$$

From Eq. (2.10), it follows that the change of the matrix $\mathcal{T}_k$ within the interval $[z_k,z_{k+1}]$ is governed by Eq. (2.9) with the boundary-value $\mathcal{T}_{1,k-1}$, $k>2$.

III. PHYSICAL ONSET OF THE COMPOUND MODEL

Here we derive an example of the compound model having application in fiber optics. The model includes the MNLSE model and the new integrable one describing the wave mixing in the resonant medium. Therefore, we first derive this new model separately and then derive the compound model. The next subsection of this section is devoted to the physical onset of the new model of resonant interaction.

A. The three-wave mixing in a resonant medium

Let us consider the three-wave mixing in the medium possessing a resonant transition. Using a set of physical assumptions, we derive from corresponding Maxwell equations the integrable system of two coupled equations.

Let the three-component field propagate in one-dimensional medium

$$\overline{E}(x,t)=\sum_{j=1}^{3}\{P_j\exp[i(q_jx-\omega_jt)]+\text{c.c.}\},\qquad(3.1)$$

here P_j is the slow changing envelope, ω_j is the carrying frequencies and q_j is the carrying wave vectors, respectively. It is assumed that the resonance conditions are the following:

$$\omega_1+\delta_1\omega_2=\omega_0+\nu,\quad \omega_3+\delta_2\omega_1=\omega_0+\nu,\qquad(3.2)$$

here, ω_0 is the frequency of transition and the frequency mismatch ν satisfies $\nu\ll\omega_i$, $j=0-3$; $\delta_{1,2}=\pm1$.

The resonance conditions (3.2) not only allow the nonlinear mixing effect to enhanced considerably (by order of magnitude), but also permit us to drop the terms in the equation that describes the cubic self interaction of the fields. This is also what allows the ISTM to be employed in the model. The processes of nonlinear mixing is determined by the two-photon induced Kerr-type nonlinearity [10–12].

The standard assumption is that the time scale of the nonlinear processes and mismatch is such that one may neglect the relaxation and adiabatically eliminate the polarization of medium from evolution equations. Substituting this in the Maxwell equations, one obtains an equation that in the rotating wave and slow changing envelopes approximations are reduced to the following system:

$$(\partial_x+v_1^{-1}\partial_t)P_1=i\frac{4\pi\omega_1}{c^2}[\alpha_{ss}P_1|P_2|^2+\alpha_{as}P_1|P_3|^2+\alpha_{pa}P_2P_1^*P_3\exp(-i\Delta x)],$$

$$(\partial_x+v_2^{-1}\partial_t)P_2=i\frac{4\pi\omega_2}{c^2}[\alpha_{ss}P_2|P_1|^2+\alpha_{sa}P_1^2P_3^*\exp(i\Delta x)],\qquad(3.3)$$

$$(\partial_x+v_3^{-1}\partial_t)P_3=i\frac{4\pi\omega_3}{c^2}[\alpha_{aa}P_3|P_1|^2+\alpha_{sa}P_2^*P_1^2\exp(i\Delta z)].$$

Here we used the resonance conditions (3.2) for $\delta_1=\delta_2=-1$. Coefficients α_{as}, etc. are the same as in Ref. [13], Chapter V. Equations (3.3) differ from that of Ref. [13] in that we neglected the relaxation processes, took into account time dependence of the fields envelopes, and used the slow amplitude approximation. $\Delta=2q_1-q_2-q_3$, Δ is the wave vectors mismatch. v_i are the phase velocities of the fields having the envelopes P_i, respectively. For other choices of δ_i, close equations can be derived.

In the majority of experiments on the observation of many-wave mixing that are familiar to us, one or more of the fields can be taken to be constant to a good approximation [12]. This condition also substantially reduces the difficulty in synchronizing the field pulses, required for the effect to be observable. The constancy of one of the fields is ensured, e.g., in the limit $|P_2|\ll|P_3|$. Let $P_3(x,t)\equiv A=\text{const.}$, therefore we withdraw the last equation from system (3.3). Under above approximations we reduce Eq. (3.3) to the system

$$\partial_\tau q_1=-2q_1^*q_2+i\nu q_1,\qquad(3.4)$$

$$\partial_z q_2=q_1^2-i2g^2q_2|q_1|^2,\qquad(3.5)$$

where

$$P_1=q_1\exp\left[-i\frac{\Delta}{2}x+ig_1\int_0^\tau\left|q_2\right|^2d\tau\right],$$

$$P_2=iq_2\exp\left[ig_1\int_0^\tau\left|q_2\right|^2d\tau\right],$$

$$g^2=-\frac{1}{2}\left(\frac{\alpha_{ss}}{\alpha_{sa}A}+\frac{1}{2}g_1\right),\quad g_1=\frac{\alpha_{ss}}{\alpha_{pa}A},$$

$$\nu=\frac{c^2\Delta}{8\pi\omega_1\alpha_{pa}}+\frac{\alpha_{as}}{\alpha_{pa}}|A|^2,$$

$$\partial_\tau=\frac{c^2}{4\pi\alpha_{pa}\omega_1}\left(\frac{\partial}{\partial x}+\frac{1}{v_1}\frac{\partial}{\partial t}\right)\quad \partial_z=\frac{c^2}{4\pi\alpha_{sa}\omega_2}\left(\frac{\partial}{\partial x}+\frac{1}{v_2}\frac{\partial}{\partial t}\right).$$

System (3.4, 3.5) is the new integrable system of equations having an application in nonlinear interaction of waves in the resonant medium. Although this model is relative to the one obtained by the author in Ref. [11], the above system (3.4, 3.5) poses as mathematical as physical features that distinguish it from known models.

B. Compound model

In experimental setup, light pulses pass as usual through different nonlinear and linear media. Let a light pulse $\mathcal{E}$ interact with two auxiliary fields $\mathcal{G},\mathcal{U}$ in the media placed in $[z_1,z_2]$ by the same way as described above. In the above notations, $\mathcal{E}\equiv P_2$, $\mathcal{G}\equiv P_3=\text{const.}$, $\mathcal{U}\equiv P_1$. The second nonlinear medium extended in the interval $[z_2,\infty)$ is an optical fiber. The field evolution in fiber is described by the MNLSE. Then the compound model is the following:

$$\partial_x\mathcal{E}+\theta(z-z_2)[iD\partial_{tt}^2\mathcal{E}+\tilde{\alpha}|\mathcal{E}|^2\mathcal{E}+\tilde{\beta}\partial_t(|\mathcal{E}|^2\mathcal{E})]=[\theta(z)\theta(z_2-z)](-2\tilde{\gamma}_1\mathcal{U}^2\mathcal{G}+2i\tilde{g}_1^2\mathcal{E}|\mathcal{U}|^2),\qquad(3.6)$$

$$[\theta(z)\theta(z_2-z)](\partial_t\mathcal{U}-\tilde{\gamma}_2\mathcal{E}U^*+2i\tilde{g}_2^2\mathcal{U}|\mathcal{E}|^2+i\tilde{\nu}\mathcal{U})=0. \tag{3.7}$$

Here $\theta(z)$ is the step function. Physical meaning of the coefficients can be found in [1].

The differentiated cubic nonlinearity in the left-hand side of Eq. (3.6) arises, then the time duration of pulses becomes compatible with the carrying frequency of the light field. The terms $\sim\tilde{g}_{1,2}^2$ arise as the main third-order terms.

Interactions of the fields in the resonant and Kerr media are spatially separated. Therefore, these interactions can be associated with the different scales. For instance, interaction in fibers can be considered as nonlinear for distance that is much more than that of resonant medium. Amplitude $\mathcal{G}$ can be taken as a constant for large enough $|\mathcal{G}|$.

Let

$$|\tilde{g}_2^2\mathcal{U}|\mathcal{E}|^2|\ll|\tilde{\nu}\mathcal{U}|. \tag{3.8}$$

After substitutions,

$$\mathcal{E}=\tilde{\rho}Q\exp(-2i\tilde{\nu}t),\quad \mathcal{U}=\tilde{\delta}U\exp(-i\tilde{\nu}t),\quad t=\tilde{\kappa}\tau, \tag{3.9}$$

the system (3.6, 3.7) can be transformed to the following form:

$$\partial_z Q-\alpha_{2,3}\{\partial_{\tau\tau}^2 Q-2i(|Q|^2-|Q^\pm|^2)Q+g^2\partial_\tau[(|Q|^2-|Q^\pm|^2)Q]\}=\alpha_{1,2}(2U^2+2ig^2Q|U|^2), \tag{3.10}$$

$$\alpha_{1,2}(\partial_\tau U+hQU^*)=0, \tag{3.11}$$

where

$$\frac{\partial}{\partial z}=\frac{1}{\zeta_0}\left(\frac{\partial}{\partial x}+\frac{1}{v_0}\frac{\partial}{\partial t}\right),\quad \tilde{\rho}=\sqrt{\frac{2\zeta_0}{\tilde{\alpha}+2\tilde{\nu}}},\quad \tilde{\kappa}=\sqrt{\frac{D}{\zeta_0}},$$

$$\frac{1}{v_0}=\frac{4\tilde{\beta}D\tilde{\nu}^2}{\tilde{\alpha}+2\tilde{\nu}}-D\tilde{\nu},\quad \tilde{\delta}^2=\frac{\sqrt{2\zeta_0}}{\tilde{\gamma}_1 G\sqrt{\tilde{\alpha}+2\tilde{\nu}}},$$

$$\zeta_0=\frac{\tilde{g}_1^2\sqrt{\tilde{\alpha}+2\tilde{\nu}}}{\tilde{\beta}\sqrt{2D}\gamma_1 G}, \tag{3.12}$$

$$|Q^\pm|^2=\frac{2D\tilde{\nu}^2}{\zeta_0},\quad g^2=\frac{2\tilde{\beta}\sqrt{\zeta_0}}{\sqrt{D}(\tilde{\alpha}+2\tilde{\nu})},\quad \widetilde{\alpha_1}(z)\equiv\tilde{\alpha}_2(z)\equiv 1.$$

Applicability of the ISTM to Eqs. (3.10, 3.11) requires that the following condition must be fulfilled

$$h=\frac{\tilde{\gamma}_2 G\sqrt{2D}}{\sqrt{\tilde{\alpha}+2\tilde{\nu}}}=1. \tag{3.13}$$

Condition (3.13) can be satisfied by choosing mismatch $\tilde{\nu}$ and (or) by choosing the amplitude G. $Q^\pm$ in Eqs. (3.10, 3.11) are determined with an accuracy to the transform $z\to\tilde{z}=z+(1-f^2)\tau/g^2$, $Q\to\tilde{Q}=Q\exp[-2i(1-f^2)z]$, $U\to\tilde{U}=U\exp[-i(1-f^2)z]$, which yields the transform $Q^\pm\to\tilde{Q}^\pm=fQ^\pm$.

IV. THE ISTM APPLICATION

To demonstrate integrability of the new model by using the ISTM we present the Lax pair of this model, which exists for any real meaning of the physical coefficients before all terms in Eqs. (3.4, 3.5). The Lax pair is

$$\partial_\tau\Phi=\begin{pmatrix}(-i\eta^2+i\nu) & 2(1-ig\eta)q_2\\ 2(1+ig\eta)\overline{q_2} & (i\eta^2-i\nu)\end{pmatrix}\Phi, \tag{4.1}$$

$$\partial_z\Phi=\frac{i}{\eta^2}\begin{pmatrix}-(1+g^2\eta^2)F_3 & -(1-ig\eta)F_+\\ (1+ig\eta)F_- & (1+g^2\eta^2)F_3\end{pmatrix}\Phi, \tag{4.2}$$

here $F_3=|q_1|^2$, $F_+=q_1^2$, $F_-=\overline{q_1^2}$, η is the spectral parameter, $\Phi(\tau,z;\lambda)$ is the matrix-valued function.

But here we will investigate the compound model (3.10, 3.11). Lax pair for this model is the following:

$$\partial_\tau\Phi=\begin{pmatrix}-i\lambda^2-ig^{-2} & \lambda q\\ \lambda\overline{q} & i\lambda^2+ig^{-2}\end{pmatrix}\Phi\equiv L_1\Phi,\quad \tau\geqslant 0; \tag{4.3}$$

$$\partial_z\Phi=\frac{i\alpha_{1,2}(z)\lambda g^3}{\lambda^2g^2+1}\begin{pmatrix}\lambda g|U|^2 & iU^2\\ i\overline{U}^2 & -\lambda g|U|^2\end{pmatrix}\Phi+\alpha_{2,3}(z)\times\begin{pmatrix}-iH_{11} & H_{12}\\ H_{21} & iH_{11}\end{pmatrix}\Phi\equiv\mathcal{A}\Phi, \tag{4.4}$$

here, λ is the spectral parameter, $q=Q/(ig)$, $q^\pm=Q^\pm/(ig)$;

$$H_{11}=[2(\lambda^2+g^{-2})^2-|q^\pm|^2(2\lambda^2+g^{-2})-\lambda^2|q|^2],$$

$$H_{12}=\lambda[2q(\lambda^2+g^{-2}-|q^\pm|^2)+i\partial_\tau q-|q|^2q], \tag{4.5}$$

$$H_{21}=\lambda[2\overline{q}(\lambda^2+g^{-2}-|q^\pm|^2)-i\partial_\tau\overline{q}-|q|^2\overline{q}].$$

This system possesses the new distinguished properties, which are interesting both for theoretical study and for application in nonlinear optics.

The ISTM formalism used here is based on the RH problem associated with the Wadati-Konno-Ichikawa (4.3) [14] linear spectral problem. Below we describe the Riemann-Hilbert problem formulation following to Refs. [15,16]. Differences are in choosing of the semi-line problem instead of the full axis spectral problem as in Refs. [15,16] and respective alteration of the Jost functions. Therefore, we describe here the Riemann-Hilbert problem without proof.

For function $q(\tau,z)$ of τ and all its derivatives decrease faster than any positive degrees of τ, $\tau\to\infty$, define the vector functions $\psi^\pm(\tau,z;\lambda)$ and $\phi^\pm(\tau,z;\lambda)$ as the Jost solutions of

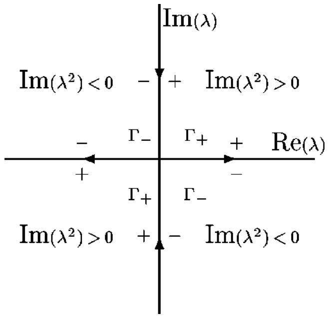

FIG. 1. Continuous spectrum $\hat{\Gamma}:\{\lambda, \mathrm{Im}\,\lambda^2=0\}$ and contours of integration $\Gamma_{\pm}$.

the first equation of system $[\partial_\tau - L_1(\tau,z;\lambda)]\psi^{\pm}(\tau,z;\lambda)=0$ and $[\partial_\tau - L_1(\tau,z;\lambda)]\phi^{\pm}(\tau,z;\lambda)=0$, with the following asymptotic,

$$\lim_{\tau\to+\infty}\psi^{+}(\tau,z;\lambda)=(0,1)^{T}e^{i\Lambda^2\tau},$$

$$\lim_{\tau\to+\infty}\psi^{-}(\tau,z;\lambda)=(1,0)^{T}e^{-i\Lambda^2\tau}, \tag{4.6}$$

$$\phi^{+}(0,z;\lambda)=(1,0)^{T},\quad \phi^{-}(0,z;\lambda)=(0,-1)^{T}, \tag{4.7}$$

where $\Lambda^2=\lambda^2+g^{-2}$, the superscripts $\pm$ mean $\mathrm{Im}(\lambda^2)\gtrless 0$, and T denotes transposition. From the Cauchy theorem follows that condition (4.7) fulfills $\forall\ \tau<0$.

The Jost solutions have the following properties:

$$\psi^{\pm}(\tau,z;\lambda)=\mp a^{\pm}(z;\lambda)\phi^{\mp}(\tau,z;\lambda) + b^{\mp}(z;\lambda)\phi^{\pm}(\tau,z;\lambda),\quad \mathrm{Im}(\lambda^2)=0, \tag{4.8}$$

where $a^{\pm}(\lambda)$ analytically expandable in $\mathrm{Im}(\lambda^2)\gtrless 0$ and for $\lambda\to\infty$, $\mathrm{Im}(\lambda^2)>0$, $a^{+}(\lambda)=1+\mathcal{O}(\lambda^{-2})$, for $\mathrm{Im}(\lambda^2)=0$, $a^{+}(\lambda)a^{-}(\lambda)+b^{+}(\lambda)b^{-}(\lambda)=1$, $a^{+}(\lambda)=\overline{a^{-}(\overline{\lambda})}$, $b^{+}(\lambda)=-\overline{b^{-}(\overline{\lambda})}$, $a^{\pm}(-\lambda)=a^{\pm}(\lambda)$, $b^{\pm}(\lambda)=-b^{\pm}(-\lambda)$, and $\rho^{\pm}(\lambda)=b^{\pm}(\lambda)/a^{\pm}(\lambda)$, and $\hat{\Gamma}=\{\lambda;\mathrm{Im}(\lambda^2)=0\}$ is the contour oriented as in Fig. 1;

$$\psi^{+}(\tau,z;\lambda)e^{-i\Lambda^2\tau}=\left(\frac{q(\tau,z)}{2\lambda},1\right)^{\mathrm{T}}+\mathcal{O}(\lambda^{-2}),$$

$$\phi^{+}(\tau,z;\lambda)e^{i\Lambda^2\tau}=\left(1,\frac{\overline{q}(\tau,z)}{2\lambda}\right)^{\mathrm{T}}+\mathcal{O}(\lambda^{-2}),$$

and, for $\lambda\to\infty$, $\mathrm{Im}(\lambda^2)<0$, $a^{-}(z;\lambda)=1+\mathcal{O}(\lambda^{-2})$,

$$\psi^{-}(\tau,z;\lambda)e^{i\Lambda^2\tau}=\left(1,\frac{\overline{q}(\tau,z)}{2\lambda}\right)^{\mathrm{T}}+\mathcal{O}(\lambda^{-2}),$$

$$\phi^{-}(\tau,z;\lambda)e^{-i\Lambda^2\tau}=-\left(\frac{q(\tau,z)}{2\lambda},1\right)^{\mathrm{T}}+\mathcal{O}(\lambda^{-2}).$$

Let $\psi(\tau,z;\lambda)\equiv\mu(\tau,z;\lambda)\exp(-i\Lambda^2\tau\sigma_3)$. Define $\mathrm{Im}(\lambda^2)>0$,

$$\mu^{+}(\tau,z;\lambda)\equiv\begin{pmatrix}\dfrac{\phi_1^{+}(\tau,z;\lambda)}{a^{+}(z;\lambda)} & \psi_1^{+}(\tau,z;\lambda)\\ \dfrac{\phi_2^{+}(\tau,z;\lambda)}{a^{+}(z;\lambda)} & \psi_2^{+}(\tau,z;\lambda)\end{pmatrix}\exp(i\Lambda^2\tau\sigma_3);$$

for $\mathrm{Im}(\lambda^2)<0$,

$$\mu^{-}(\tau,z;\lambda)\equiv\begin{pmatrix}\psi_1^{-}(\tau,z;\lambda) & -\dfrac{\phi_1^{-}(\tau,z;\lambda)}{a^{-}(z;\lambda)}\\ \psi_2^{-}(\tau,z;\lambda) & -\dfrac{\phi_2^{-}(\tau,z;\lambda)}{a^{-}(z;\lambda)}\end{pmatrix}\exp(i\Lambda^2\tau\sigma_3);$$

and $\rho(\lambda)\equiv\rho^{-}(\lambda)$. Then 2×2 matrix function $\mu(\tau,z;\lambda)$ $[\det\{\mu(\tau,z;\lambda)\}=1]$ solves the following Riemann-Hilbert problem:

(1) $\mu(\tau,z;\lambda)$ is holomorphic $\forall\ \lambda\in\mathbb{C}-\hat{\Gamma}$.

(2) $\mu(\tau,z;\lambda)$ satisfies the following jump conditions:

$$\mu^{+}(\tau,z;\lambda)=\mu^{-}(\tau,z;\lambda)E(\tau;\lambda)^{-1}G(\lambda)E(\tau;\lambda),\quad \lambda\in\hat{\Gamma}, \tag{4.9}$$

where

$$G(\lambda)=\begin{pmatrix}1-\overline{\rho(\overline{\lambda})}\rho(\lambda) & \rho(\lambda)\\ -\overline{\rho(\overline{\lambda})} & 1\end{pmatrix},$$

$$E(\tau;\lambda)=\exp(-i\Lambda^2\tau\sigma_3),$$

$\rho(\lambda)\in\mathcal{S}(\hat{\Gamma})$, and $\rho(-\lambda)=-\rho(\lambda)$.

(3) For $\lambda\to\infty$, $\lambda\in\mathbb{C}-\hat{\Gamma}$,

$$\mu(\tau,z;\lambda)=\mathrm{I}+\mathcal{O}(\lambda^{-1}).$$

Above properties follow from definition of $\mu(\tau,z;\lambda)$ for $\mathrm{Im}(\lambda^2)\gtrless 0$ and analytical properties of the Jost functions.

Let $||\rho||_{\mathcal{L}^{\infty}(\hat{\Gamma})}\equiv\sup_{\lambda\in\hat{\Gamma}}|\rho(\lambda)|<1$, [17,18]. Then: (i) the RH problem is uniquely solvable; (ii) $\psi(\tau,z;\lambda)\equiv\mu(\tau,z;\lambda)\exp(-i\Lambda^2\tau\sigma_3)$ is the solution of system (4.3,4.4);

$$q(\tau,z)\equiv 2i\lim_{\lambda\to\infty}[\lambda\mu(\tau,z;\lambda)]_{12}$$

and

$$\overline{q}(\tau,z)\equiv -2i\lim_{\lambda\to\infty}[\lambda\mu(\tau,z;\lambda)]_{21}; \tag{4.10}$$

and $\mu(\tau,z;\lambda)$ possesses the following symmetry reductions, $\mu(\tau,z;-\lambda)=\sigma_3\mu(\tau,z;\lambda)\sigma_3$ and $\mu(\tau,z;\lambda)=\sigma_1\overline{\mu(\tau,z;\overline{\lambda})}\sigma_1$.

The solvability of the RH problem $\forall\ z$ follows from the condition $||\rho||_{\mathcal{L}^\infty(\hat{\Gamma})}<1$, see details in Refs. [16–19]. Condition (ii) follows from Eq. (4.3) and decomposition on λ^{-1}, $\lambda\to\infty$.

For $a(\lambda)\neq 0$, assuming that $a(\lambda)\neq 0$, the first vector of the above equation implies (taking the complex conjugate and the plus projection) the integral equations

$$\overline{\psi_1^+}(\tau,z;\lambda)=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2i\pi}\int_{\Gamma_+}\overline{\rho}(z;\lambda)e^{2i\tau\Upsilon^2}\overline{\psi_2^-}(\tau,z;\lambda)\frac{d\zeta}{\zeta-\lambda}, \tag{4.11}$$

$$\overline{\psi_2^-}(\tau,z;\lambda)=\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{2i\pi}\int_{\Gamma_-}\rho(z;\lambda)e^{-2i\tau\Upsilon^2}\overline{\psi_1^+}(\tau,z;\lambda)\frac{d\zeta}{\zeta-\lambda}. \tag{4.12}$$

Here, $\Upsilon^2=\zeta^2+g^{-2}$. Contours $\Gamma_\pm$ oriented as in Fig. 1 include the paths along the axis and arcs in infinity. The formulas (4.11, 4.12) taking into account (4.10) and dependence $\rho=\overline{b}/a$, $\overline{\rho}=b/a$ on z, furnish the solution of the inverse problem.

V. QUASIRADIATION SOLUTION TO THE COMPOUND MODEL

In this section we show that a quasiradiation solution to system (3.10, 3.11) for the field $Q(\tau,z)$ is generated by a simple ''boundary'' ($\tau=0$) conditions for the auxiliary field U.

Impose the following initial-boundary conditions:

$$Q(\tau,0)=0,\quad \tau\geqslant 0,\quad Q(0,z)=0,\quad \forall\ z;$$

$$U(z,0)=U_0=\text{const.}\neq 0,\quad z\in[0,z_2];\quad U(z,0)=0,\quad z>z_2. \tag{5.1}$$

It can be shown that a trivial solution $Q(\tau,z)\equiv 0$ to Eq. (3.10) for $z>z_2$ is linearly stable. This fact and conditions (5.1) justify the following assumption: $Q(\tau,z)=0$, $\tau\to\infty$; $\forall\ z$.

From the symmetry properties of spectral problem (4.3) follows that matrix T has the form

$$T(\lambda)=\begin{pmatrix}a(\lambda) & \overline{b(\lambda)}\\ b(\lambda) & \overline{a(\lambda)}\end{pmatrix}. \tag{5.2}$$

We search for the solution of Eq. (2.7) for 2×2 matrix $\mathcal{A}(0,z;\lambda)$: $\det\mathcal{A}(0,z;\lambda)\neq 0$, $\forall\ z$, which is constant within intervals $A_1(0,z;\lambda)\equiv A_1(0,0;\lambda)$, $z\in[0,z_2]$; $A_2(0,z;\lambda)\equiv A_2(0,\infty;\lambda)$, $z>z_2$. $\tilde{\alpha}_i(z)\equiv 1, i=1,2$. $\mathcal{A}(0,z;\lambda)$ changes stepwise in the point $z=z_2$. For the initial-boundary conditions (5.1) this solution $\forall\ z>0$ is

$$a(z;\lambda)=\frac{a_0}{2\Omega}\{[(\Omega+\tilde{A}_{11})+\epsilon\rho_0\tilde{A}_{21}]e^{-[z\Omega+\Theta(z)]}+[(\Omega-\tilde{A}_{11})-\epsilon\rho_0\tilde{A}_{21}]e^{[z\Omega-\Theta(z)]}\}, \tag{5.3}$$

$$\overline{b}(z;\lambda)=\frac{a_0}{2\Omega}\{[(\Omega-\tilde{A}_{11})\rho_0+\epsilon\tilde{A}_{12}]e^{-[z\Omega+\Theta(z)]}+[(\Omega+\tilde{A}_{11})\rho_0-\epsilon\tilde{A}_{12}]e^{[z\Omega-\Theta(z)]}\}, \tag{5.4}$$

where $\tilde{A}_{ij}$ is the element of the matrix $\tilde{A}=\Phi_0^{-1}(\lambda)\mathcal{A}(0,0;\lambda)\Phi_0(\lambda)=\tilde{A}(0,0;\lambda)$, $\Theta(z;\lambda)=\int_0^z\tilde{A}_{11}^+(s;\lambda)ds$, $\tilde{A}_{11}^+(z;\lambda)=\lim_{\tau\to\infty}[\Phi_+^{-1}(\lambda)\mathcal{A}(\tau,z;\lambda)\Phi_+(\lambda)]_{11}=\lim_{\tau\to\infty}\mathcal{A}_{11}(\tau,z;\lambda)$, $\Omega^2=\tilde{A}_{11}^2(0,z;\lambda)+\tilde{A}_{12}(0,z;\lambda)\tilde{A}_{21}(0,z;\lambda)$, $\rho_0(\lambda)=\rho(z=0,\lambda)=\overline{b}_0/a_0$.

The coefficient $\rho(z;\lambda)$ is

$$\rho(z;\lambda)=\frac{\overline{b}}{a}=\frac{[(\Omega-\tilde{A}_{11})\rho_0+\epsilon\tilde{A}_{12}]e^{-2\Omega z}+[(\Omega+\tilde{A}_{11})\rho_0-\epsilon\tilde{A}_{12}]}{[(\Omega+\tilde{A}_{11})+\epsilon\rho_0\tilde{A}_{21}]e^{-2\Omega z}+[(\Omega-\tilde{A}_{11})-\epsilon\rho_0\tilde{A}_{21}]}. \tag{5.5}$$

Coefficient $\rho_0(\lambda)$ is determined by a solution of the spectral problem (4.3) for a fixed potential $q(\tau,0)$.

The matrix $T\ \forall\ z$ is

$$T(z;\lambda)=s_0\begin{pmatrix}\left[\coth(\Omega\tau)-\frac{\tilde{A}_{11}}{\Omega}+\rho_0\frac{\tilde{A}_{21}}{\Omega}\right]e^{i\Theta} & \left\{-\frac{\tilde{A}_{12}}{\Omega}+\rho_0\left[\coth(\Omega\tau)+\frac{\tilde{A}_{11}}{\Omega}\right]\right\}e^{i\Theta}\\ \left\{\overline{\rho_0}\left[\coth(\Omega\tau)-\frac{\tilde{A}_{11}}{\Omega}\right]-\frac{\tilde{A}_{21}}{\Omega}\right\}e^{-i\Theta} & \left\{\left[\coth(\Omega\tau)+\frac{\tilde{A}_{11}}{\Omega}\right]-\overline{\rho_0}\frac{\tilde{A}_{12}}{\Omega}\right\}e^{-i\Theta}\end{pmatrix}, \tag{5.6}$$

where $s_0=a_0\sinh(\Omega\tau)$,

$$\Omega=\frac{\mathrm{i}}{\Lambda^2}\{\alpha_{1,2}(z)[(1-g^2\Lambda^2)|U_0|^2+\alpha_{2,3}(z)2\Lambda^6]^2-\alpha_{1,2}(z)\times(1-g^2\Lambda^2)|U_0|^4\}^{1/2},$$

$$\tilde{\mathcal{A}}_{11}=A_{11}=-\alpha_{1,2}(z)i|U_0|^2\frac{1-g^2\Lambda^2}{\Lambda^2}-2\alpha_{2,3}(z)\mathrm{i}\Lambda^4,$$

$\Lambda^2=\lambda^2+g^{-2}$, $\widetilde{\alpha_i}(z)\equiv 1$. Expression for $\tilde{\mathcal{A}}_{12}$ ($\tilde{\mathcal{A}}_{21}$) will not be used below and we omit it.

Taking into account that $\mathcal{A}=\alpha_{1,2}(z)A_1+\alpha_{2,3}(z)A_2=A_2$ for $z>z_2$ and using Eq. (5.6), it can easily shown that for conditions (5.1) the following relation is fulfilled

$$T(z;\lambda)=T_+(z_2,z)T(z_2;\lambda)T_+^{-1}(z_2,z),\quad z>z_2,\tag{5.7}$$

here $T_+(z_2,z)$ is the formal solution to the linear system (4.4) for zero nondiagonal elements of matrix $A(\tau_2,z;\lambda)$

$$T_+(z_2,z;\lambda)=e^{-i2\sigma_3\Lambda^4z},\quad z>z_2.\tag{5.8}$$

From Eq. (5.7) follows

$$a(z,\lambda)=a_2e^{-\Theta},\quad \overline{b(z,\lambda)}=\overline{b_2}e^{(4i\Lambda^4z-\Theta)},\tag{5.9}$$

here a_2, $\overline{b_2}$ are the elements of matrix $T(0,z_2,\lambda)$.

To solve Eqs. (4.11, 4.12) one must find $\rho(z;\lambda)$ and the zeros of $a(\lambda)$. Poles position in complex plane is derived from the equation $T_{11}(z;\lambda)=a(z;\lambda)=0$, i.e.,

$$a_0\left[\coth(\Omega\tau)-\frac{\tilde{A}_{11}}{\Omega}\right]+\overline{b_0}\frac{\tilde{A}_{21}}{\Omega}=0.\tag{5.10}$$

From condition $q(\tau,0)\equiv 0$, it follows that $b_0(z;\lambda)=0$. Then the poles are determined by the equation

$$\coth(\Omega z)-\frac{\tilde{\mathcal{A}}_{11}}{\Omega_u}=0.\tag{5.11}$$

For $z=z_2$ we get can rewrite Eq. (5.11) in the form

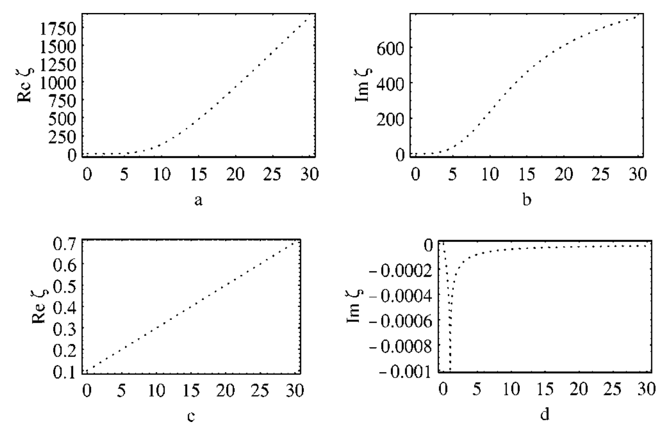

FIG. 2. Dependence of the imaginary and the real parts of ζ on n. Figs. (a,b) are depicted for $L_0=0.01$, Figs. (c,d) correspond to $L_0=50$.

$$\coth(\mathrm{i}\zeta L_0)=\zeta,\tag{5.12}$$

here, $\zeta=\pm\sqrt{g^2\Lambda^2-1}/g\Lambda$, $L_0=z_2|U_0|^2g^2$. This equation has a set of solutions $\zeta(n)$, n is an integer, which satisfy to

$$\zeta(n)L_0+i\ln\frac{\zeta(n)+1}{\zeta(n)-1}=\frac{\pi n}{L_0}.\tag{5.13}$$

Numerical solution to Eq. (5.13) is depicted in Fig. 2. We reveal numerically that for $L_0\ll 1$ $\mathrm{Im}\,\zeta(n)$ tends to nonzero constant as $n\to\infty$. If $L_0\gg 1$, then one can reckon

$$\mathrm{Im}\,\zeta(n)=0,\quad \mathrm{Re}\,\zeta(n)=n\pi/L_0,\quad n=0,\pm 1,\pm 2,\pm 3,\ldots,\tag{5.14}$$

with an accuracy $\mathcal{O}(1/(L_0\log L_0))$.

Solutions to (4.11, 4.12) for $z\in[0,z_2]$ describe the fields dynamics in the resonant medium, where the dependence ρ on z yields an infinite number of poles (5.13). For $z>z_2$ (optical fiber) the dependence of ρ on z becomes simple

$$\rho(z;\lambda)=\rho(z_2)e^{4i(\lambda^2+g^{-2})^2z}=\rho(z_2)e^{4i\Lambda^4z}.\tag{5.15}$$

Therefore to solve Eqs. (4.11, 4.12) we have to calculate all residues in $\lambda_n:\lambda_n^2=g^{-2}\zeta^2(n)[1-\zeta^2(n)]^{-1}$ in the right-hand side (RHS) of equation (4.11),

$$F_1(\tau,z_2;\xi)=\sum_{n=-\infty}^{\infty}\frac{e^{-i\Lambda_n^2\tau+4i\Lambda_n^4z\theta(z-z_2)}}{\xi-\lambda_n}\cdot\frac{c_1(\lambda_n)+c_2(\lambda_n)e^{2iL\Omega(\lambda_n)}}{c_4(\lambda_n)e^{2iL\Omega(\lambda_n)}2iL\Omega'(\lambda_n)}\psi_2(\tau;\lambda_n),\tag{5.16}$$

here $\Omega'(\lambda_n)=\lim_{\lambda\to\lambda_n}\partial/\partial\lambda\Omega(\lambda)$. $e^{2iL\Omega(\lambda_n)}=-c_3(\lambda_n)/c_4(\lambda_n)$. $\Lambda_n^2=\lambda_n^2+g^{-2}$. Coefficients c_i, $i=1-4$ for $b_0(z;\lambda)=0$ are: $c_1(\lambda)=\epsilon A_{12}=-c_2(\lambda)$, $c_3(\lambda)=\Omega+A_{11}$, $c_4(\lambda)=\Omega-A_{11}$, see, Eq. (5.5). In Eq. (5.16) the matrix elements $A_{ij}(z)\equiv A_{ij}(0)$ are taken in the interval $[0,z_2]$. The first multipliers of the terms of sum (5.16) are associated with the evolution on z within interval the $[z_2,\infty)$. The second multipliers of such a term originate from z dependence of scattering data within the first interval $[0,z_2]$ and therefore are associated with the evolution of fields during the wave mixing in the resonant medium.

Below we restrict our investigation to the case of large effective length of the resonant medium, i.e., $L_0\gg 1$. Let us substitute λ_n in Eq. (5.16) and use Eq. (5.14). For $L_0\gg 1$ coefficients in the sum (5.16) change smoothly and difference between the elements of sum tends to zero for $L_0\to\infty$.

These statements justify the transformation from summation in Eq. (5.16) over n to integration with respect to $\mu = n\pi/L_0$

$$F_1(\tau,z;\xi)=\int_{\Gamma_+}\frac{e^{2iX^2\tau+4iX^4z\theta(z-z_2)}}{\xi-\lambda}\frac{\psi_1(\tau;\mu)g\mu d\mu}{\pi(1-\mu^2)^3}$$
$$=\frac{g^3}{2\pi}\int_{\Gamma_+}\frac{e^{2iX^2\tau+4iX^4z\theta(z-z_2)}}{\xi-\chi}\psi_1(\tau;\chi)dX^2; \tag{5.17}$$

where $X^2=g^{-2}(1-\mu^2)^{-2}+g^{-2}=\chi^2+g^{-2}$. Then we compute the integral in the RHS of Eq. (4.12) by the same way.

As a result, we obtain the following integral equations:

$$\overline{\psi_1^+}(\tau,y;\lambda)=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2i\pi}\int_{\Gamma_+}\overline{\rho_{eff}}(\chi)e^{2i\tau X^2+4iyX^4}\overline{\psi_2^-}(\tau,y;\chi)\frac{d\chi}{\lambda-\chi}, \tag{5.18}$$

$$\overline{\psi_2^-}(\tau,y;\lambda)=\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{2i\pi}\int_{\Gamma_-}\rho_{eff}(\chi)e^{-2i\tau X^2-4iyX^4}\overline{\psi_1^+}(\tau,y;\chi)\frac{d\chi}{\lambda-\chi}, \tag{5.19}$$

here, $\rho_{eff}(\chi)=2i\ \ g^3\chi,\ X^2=\chi^2+g^{-2},\ y=z-z_2>0$.

Systems (5.18) and (5.19) describe a radiation solution to the MNLSE associated with the continuous spectrum $\hat{\Gamma}:\mathrm{Im}\,\lambda^2=0$ and the effective coefficient ρ_{eff}.

An asymptotic solution to Eqs. (5.18) and (5.19) may be derived by using sophisticated techniques, developed in Refs. [15,18,19].

Let us find a solution for a small τ. Using Eq. (4.10), we find for $(\overline{\psi_1^+})_2\approx 1,\ (\overline{\psi_2^-})_2\approx 0$,

$$q(\tau,y)\approx\frac{\epsilon\sqrt{i}g^3}{2\pi\sqrt{2y}}e^{\frac{i\tau^2}{8y}}\left[D_{-1}\left(\frac{\sqrt{i}\tau}{\sqrt{2y}}\right)+D_{-1}\left(-\frac{\sqrt{i}\tau}{\sqrt{2y}}\right)\right],$$
$$y=z-z_2>0, \tag{5.20}$$

here $D_{-1}(y)$ is the function of the parabolic cylinder [20].

VI. CONCLUSION

Application of compound integrable models in nonlinear optics is conditioned by the fact that experimental setup includes different linear and nonlinear media as usual. Light pulses can be amplified, squeezed, and deformed while they propagate through these media. From another side the compound models extend the region of application of the ISTM.

It have been shown in this paper that the quasiradiation solution to the MNLSE in one medium can be generated by the simple boundary conditions (at $\tau=0$) for the field interacting in an another medium. Analogous results can be derived for a more simple case of the model including the Maxwell-Bloch model and the nonlinear Schrödinger equation (NSE) compounded by the same way as considered here. It can be shown that for analogous initial-boundary conditions a quasiradiation solution to the NSE is generated by the nonzero initial polarization of medium. These results indicate that asymptotic behavior of the solution found in the present paper is expected to occur in other compound models.

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